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Guidance of a Homing Missile via Nonlinear Geometric Control Methods

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This paper examines the guidance problem of an acceleration constrained homing missile when the initial missile heading is far from intercept course. A guidance strategy based on the theory of feedback linearization is presented, and simulation results are given comparing miss distance performance of the feedback linearized guidance law to proportional navigation. It is demonstrated that the feedback linearized guidance law is a viable option under these conditions.

Nomenclature

$A_{\max}$	= missile's acceleration limit
a_m	= missile's acceleration perpendicular to missile's velocity vector
a_T	= target's acceleration perpendicular to target's velocity vector
C^∞	= set of all functions that are infinitely differentiable
(K_1, K_2)	= constant gains for feedback linearized guidance law
$L_f^k h(x)$	= kth Lie derivative of $h(x)$ with respect to $f(x)$
$L_g L_f^k h(x)$	= Lie derivative of $L_f^k h(x)$ with respect to $g(x)$
N'	= proportional navigation's effective navigation ratio
$\Re^n$	= n-dimensional Cartesian space
R_{TM}	= target to missile range
V_c	= velocity at which the missile closes on the target
$(V_{c_{D1}}, V_{c_{D2}})$	= desired closing velocities
V_m	= magnitude of the missile's velocity vector
V_T	= magnitude of the target's velocity vector
(V_{x_m}, V_{y_m})	= missile's velocity in inertial frame
(V_{x_T}, V_{y_T})	= target's velocity in inertial frame
(X_m, Y_m)	= missile's position in inertial frame
(X_T, Y_T)	= target's position in inertial frame
α_0	= missile's switching angle
$\Delta\Psi_m$	= missile's heading error
σ	= line-of-sight angle to the target
v	= external input to the feedback linearized system
Ψ_L	= missile's look angle
Ψ_m	= missile's heading direction
$\Psi_{m_{\mathrm{pip}}}$	= missile's predicted intercept heading direction
Ψ_T	= target's heading direction

I. Introduction

PROPORTIONAL navigation (PN) is a well-known guidance law[1–17] that performs very well in a large variety of cases. Its first derivation was based on the heuristic that zeroing the line-of-sight (LOS) rate between the missile and target must eventually lead to intercept. Later[2,7] it was shown that PN is, in fact, the optimal guidance law under the simplifying assumptions of constant missile velocity, instantaneous missile response, a nonmaneuvering target, and intercepts close to the collision course. When these assumptions do not hold, PN is no longer optimal and its performance deteriorates accordingly.

Recently, differential geometric control methods[18,19] have led to a new class of control techniques for nonlinear systems which are linear (affine) in the control inputs. The primary technique known as feedback linearization uses nonlinear feedback to cancel out any nonlinearities in the input-output behavior of the plant. Once the closed-loop system is linearized then all of the tools from the theory of linear systems can be directly applied to control the system. This method allows one to design an exact controller for the class of nonlinear systems affine in the control. To date, the feedback linearization method has been applied to derive the exact command to the line-of-sight guidance law[20] and to landing problems in Ref. 21.

In this paper, we apply the feedback linearization method to derive a guidance law for a homing missile. This enables one to construct an exact guidance law which does not require any of the standard assumptions or approximations in its formulation. The paper is organized as follows. In Sec. II a model of the geometry, kinematics, and dynamics of the system is provided which is affine in the command input, the missile's acceleration. Section III presents the closed-loop system after the linearizing feedback is applied and discusses the overall nature of the feedback linearized system. Section IV presents a guidance law for the control of the feedback linearized system. Finally, in Sec. V simulations are presented comparing the performance of the feedback linearizing guidance law against the performance of proportional navigation. The performance index is the miss distance. Two appendices are also provided containing background on the theory of feedback linearization and a detailed derivation of the feedback linearizing guidance law.

II. Problem Formulation

Figure 1 shows the intercept geometry of a missile in planar pursuit of a target. The variables pertaining to the missile are defined as follows: (X_m, Y_m) denotes the instantaneous position of the missile; (Ψ_m, V_m) denote the heading and speed of the missile with respect to a constant reference direction, respectively; and a_m denotes the missile's acceleration which is assumed to act in a direction orthogonal to the missile's instantaneous velocity vector. Similarly, the variables pertaining to the target are defined as follows: (X_T, Y_T) denotes the instantaneous position of the target; (Ψ_T, V_T) denote the heading and speed of the target, respectively with respect to the same constant reference direction already stated. Let a_T denote the target's acceleration which is also assumed to be acting in a direction

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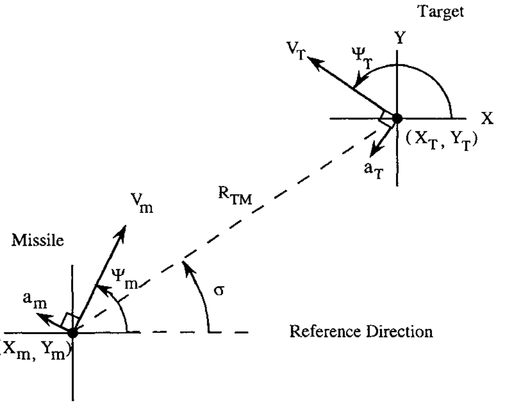

Fig. 1 Planar intercept geometry.

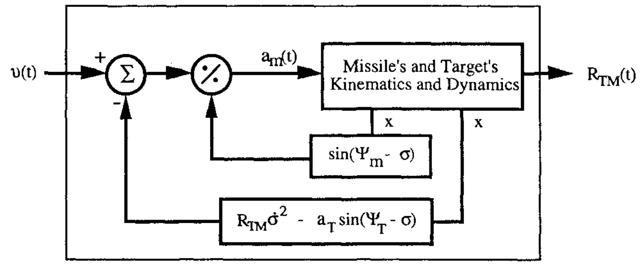

Fig. 2 Application of feedback linearization.

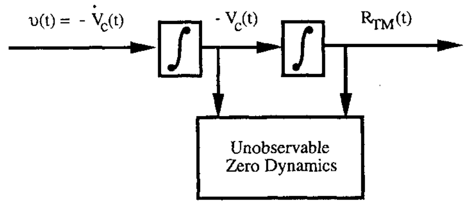

Fig. 3 Equivalent feedback linearized system.

orthogonal to the target's instantaneous velocity vector. The other variables that appear in Fig. 1 are the LOS angle σ, the angle between the constant reference direction and the LOS, and the target to missile range R_{TM}.

The planar two-dimensional missile and target kinematics can be expressed in a right-hand Cartesian coordinate system where the origin is taken to be the initial position of the missile, and the X axis, the constant reference direction, is taken to be the initial direction of the LOS. The kinematic equations expressed in this coordinate frame are

$$\ddot{x}_m = -a_m \sin\Psi_m, \qquad x_m(t_0) = x_{m_0}, \qquad \dot{x}_m(t_0) = \dot{x}_{m_0} \tag{1}$$

$$\ddot{y}_m = a_m \cos\Psi_m, \qquad y_m(t_0) = y_{m_0}, \qquad \dot{y}_m(t_0) = \dot{y}_{m_0} \tag{2}$$

$$\ddot{x}_T = -a_T \sin\Psi_T, \qquad x_T(t_0) = x_{T_0}, \qquad \dot{x}_T(t_0) = \dot{x}_{T_0} \tag{3}$$

$$\ddot{y}_T = a_T \cos\Psi_T, \qquad y_T(t_0) = y_{T_0}, \qquad \dot{y}_T(t_0) = \dot{y}_{T_0} \tag{4}$$

where

$$\Psi_m = \tan^{-1}\frac{\dot{y}_m}{\dot{x}_m}, \qquad \text{and} \qquad \Psi_T = \tan^{-1}\frac{\dot{y}_T}{\dot{x}_T}$$

The subscripts m and T refer to the missile and target variables, respectively.

Since the goal of the missile guidance law is to command the missile to intercept the target, i.e., null the target to missile range R_{TM}, the natural input for the system is the missile acceleration a_m, and the natural output of the system is the target-to-missile range,

$$R_{TM}(t) = \sqrt{(x_m(t) - x_T(t))^2 + (y_m(t) - y_T(t))^2} \geq 0 \tag{5}$$

where the initial range is $R_{TM}(t_0) = R_{TM_0}$.

The problem considered in this paper is to find $\{a_m(t): 0 < t < t_f\}$ with t_f being an unspecified final time such that $R_{TM}(t_f)$ is below a specified threshold subject to the constraint $|a_m(t)| \leq A_{max}$, where A_{max} is the missile's acceleration constraint. We assume instantaneous missile response, exact knowledge of the state, and no seeker look-angle constraints. Including additional dynamics, for example, autopilot dynamics, would significantly increase the computational complexity of the control law. State estimation for this problem via extended Kalman filtering has been explored in Refs. 22 and 23. A significant constraint on the seeker look angle will undoubtly be a key limiting factor in the performance of any homing controller.

III. Feedback Linearization

Equations (1–5) give a representation of the system in terms of a set of nonlinear differential equations which is affine in the control variable $a_m(t)$. Hence, the system can be modeled as an eighth-order state-space system of the form

$$\dot{x} = f(x) + g(x)a_m \qquad R_{TM} = h(x) \tag{6}$$

where the state variable x is defined in Appendix B. Once the system is in the form of Eq. (6), we can apply the standard tools of geometric nonlinear control theory (see Appendix A) to find the linearizing feedback

$$a_m = \frac{1}{\sin(\Psi_m - \sigma)}\left[\upsilon - R_{TM}\dot{\sigma}^2 + a_T \sin(\Psi_T - \sigma)\right] \tag{7}$$

where υ is the external input to the feedback linearized system (see Appendix B). Note that three functions of the state variables are required by the feedback linearization law, specifically, $R_{TM}\dot{\sigma}^2$, $\sin(\Psi_m - \sigma)$, and $a_T \sin(\Psi_T - \sigma)$. Also note that the control input remains bounded except when the missile's heading is directly on the line of sight, i.e., $\Psi_m = \sigma$. This uniquely corresponds to those points in the state space where the relative degree of Eq. (6) is not defined. It should also be mentioned that when the equations of relative motion are expressed in polar coordinates, the required linearizing feedback law in Eq. (7) becomes more obvious.

The resulting feedback linearized input-output system is the second-order linear system

$$\ddot{R}_{TM} = -\dot{V}_c = \upsilon \tag{8}$$

This system can be viewed as the projection of the two-dimensional kinematics and dynamics onto the one-dimensional LOS coordinate axis. Hence, the feedback linearized system preserves only the relative position, velocity, and acceleration components of the original system in the LOS direction. It contains no information about the original system in the direction perpendicular to the LOS. Figure 2 shows the actual feedback linearized input-output system. But since the closed-loop input-output system is equivalently described by Eq. (8), it can also be represented as the system shown in Fig. 3. The internal zero dynamics in this case correspond to a sixth-order nonlinear subsystem which has been made unobservable at the output by the feedback linearizing control loop. These dynamics are not apriori guaranteed to be stable. But since the homing problem is a finite time control problem, and in practice the missile's acceleration is constrained, the zero dynamics will always remain bounded during intercept. In applications, however, boundedness may not be sufficient to assure acceptable performance unless the internal variables lie within a specific dynamic range. This issue was not addressed in this paper.

IV. Synthesis of Guidance Strategies

In this section we construct a guidance law to steer the missile to the predicted intercept point. We call this the feedback linearizing guidance law (FLGL). The general idea behind predicted intercept point guidance[1] is to command the missile's heading to a direction ahead of the target such that the LOS rate $\dot{\sigma}$ is nulled exactly as in PN guidance. When this heading is achieved the missile is said to be on the heading of predicted intercept (see Fig. 4).

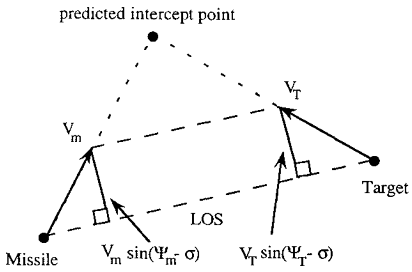

Fig. 4 Intercept triangle.

The FLGL generates the commanded acceleration to the missile as follows:

$$a_m = \begin{cases} \dfrac{1}{\sin(\Psi_m - \sigma)}\left[\upsilon - R_{\mathrm{TM}}\dot{\sigma}^2 + a_T \sin(\Psi_T - \sigma)\right], & |\Delta\Psi_m| > \alpha_0 \quad (9a) \\ N'V_c\dot{\sigma}, & |\Delta\Psi_m| \le \alpha_0 \quad (9b) \end{cases}$$

where the input to the feedback linearized system is

$$\upsilon = \begin{cases} K_1(V_c - V_{c_{D1}}), & \sin(\Psi_m - \sigma)\sin(\Psi_T - \sigma) \ge 0 \quad (9c) \\ K_2(V_c - V_{c_{D2}}), & \sin(\Psi_m - \sigma)\sin(\Psi_T - \sigma) < 0 \quad (9d) \end{cases}$$

and where

$$\Delta\Psi_m \triangleq \Psi_m - \Psi_{m_{\mathrm{pip}}}, \tag{9e}$$

$$\Psi_{m_{\mathrm{pip}}} = \sigma + \sin^{-1}\left(\frac{V_T \sin(\Psi_T - \sigma)}{V_m}\right) \tag{9f}$$

$$V_{c_{D1}} = \sqrt{V_m^2 - \{V_T \sin(\Psi_T - \sigma)\}^2} - V_T \cos(\Psi_T - \sigma) \tag{9g}$$

$$V_{c_{D2}} = V_m - V_T \cos(\Psi_T - \sigma) \tag{9h}$$

In addition,

$$V_m = \sqrt{V_{xm}^2 + V_{ym}^2}, \qquad \text{and} \qquad V_T = \sqrt{V_{xT}^2 + V_{yT}^2}$$

The rationale for using the preceding guidance law is as follows. When the difference between the actual missile heading and the predicted intercept heading $\Delta\Psi_m$ is within a neighborhood of the switching angle $+/-\alpha_0$, proportional navigation, Eq. (9b), is used to command the missile's acceleration. This is a logical strategy since PN is the optimal guidance law when the missile's heading is close to the predicted intercept heading.[2] However, when the missile's heading error is greater than $+/-\alpha_0$, the missile's acceleration is commanded through the linearizing feedback by means of the external input $\upsilon(t)$, as seen in Eq. (9a). Note that if $\alpha_0 > \sin^{-1}(V_T/V_m)$ then the relative degree problem (division by zero) is circumvented since PN will always be commanding the missile's acceleration whenever the missile's heading is pointing in the LOS direction.

Since the feedback linearized system retains only state information in the LOS direction, the feedback linearized system cannot distinguish whether the missile's heading is leading or lagging the target position (see Fig. 5). Therefore, it is necessary to devise a separate linear guidance strategy for each of the two cases.

Case 1: Missile's Velocity Vector Leading the Target

In this case the sign of the expression $\sin(\Psi_m - \sigma)\sin(\Psi_T - \sigma)$ is positive and, thus, we guide the missile with a linear state feedback given by Eq. (9c). Here we use the fact that on the predicted intercept heading the missile's velocity component orthogonal to the LOS is equal to the target's velocity component orthogonal to the LOS. Under this condition we find the expression for the desired magnitude of the closing velocity $V_{c_{D1}}$ to be given by Eq. (9g). In this equation, the square root term represents the desired magnitude of the missile's velocity in the LOS direction, whereas the remaining term $V_T \cos(\Psi_T - \sigma)$ is the magnitude of the target velocity in the LOS direction. Figure 5 shows the missile's commanded heading rotation when this linear state feedback is used. Observe that when the missile's velocity vector is lagging the target this state feedback law steers the missile to the incorrect symmetric missile heading. In this case, a different feedback strategy has to be applied.

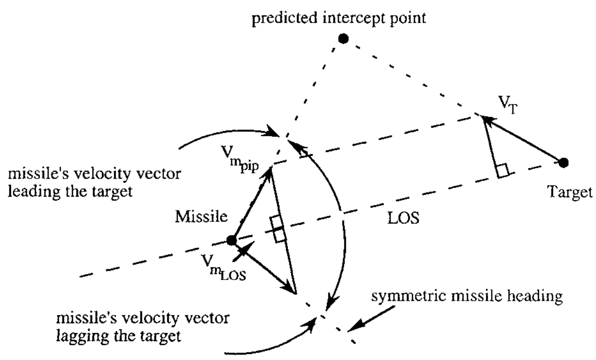

Fig. 5 Predicted intercept guidance, the missile's velocity vector rotation.

Case 2: Missile's Velocity Vector Lagging the Target

In this case the sign of the expression $\sin(\Psi_m - \sigma)\sin(\Psi_T - \sigma)$ is negative and, thus, we desire to rotate the missile's heading toward the LOS. The linear state feedback used in this case is given by Eq. (9d), where the expression for the desired magnitude of the closing velocity $V_{c_{D2}}$ is given by Eq. (9h). In Eq. (9h), V_m is the desired magnitude of the missile's velocity in the LOS direction, and $V_T \cos(\Psi_T - \sigma)$ is the magnitude of the target's velocity in the LOS direction. One of three things will happen to terminate the use of this law.

1) The law will command the missile such that the heading error of the missile $\Delta\Psi_m$ will be small enough to switch over to PN guidance.

2) The sign of the expression $\sin(\Psi_m - \sigma)\sin(\Psi_T - \sigma)$ will go positive before the missile's heading ever reaches the LOS. This results in the control of missile acceleration being switched over to the case 1 linear state feedback expression shown in Eq. (9c).

3) The missile heading is steered into a small neighborhood of the LOS, i.e., $\Psi_m \approx \sigma$, where the linearizing feedback tends to infinity [see Eq. (9a)]. This forces the missile's acceleration into saturation, $a_m(t) = +/-A_{\max}$, which causes the missile's heading to rotate to the other side of the LOS where the sign of the expression $\sin(\Psi_m - \sigma)\sin(\Psi_T - \sigma)$ becomes positive. When this sign change occurs the control of missile is again switched over to case 1 linear state feedback shown in Eq. (9c). Note that if $\alpha_0 > \sin^{-1}(V_T/V_m)$ this situation never occurs.

The implementation of the feedback linearizing guidance law requires knowledge of the following functions of state variables of the system: R_{TM}, V_c, σ, $\dot{\sigma}$, V_m, $\sin(\Psi_m - \sigma)$, $V_T \cos(\Psi_T - \sigma)$, $V_T \sin(\Psi_T - \sigma)$, and $a_T \sin(\Psi_T - \sigma)$. Normally, we can measure or derive characteristics related to the missile: R_{TM}, V_c, σ, $\dot{\sigma}$, Ψ_m, and V_m. The functions (R_{TM}, V_c) are available from the pulse-Doppler seeker.[3] The missile velocity and attitude (Ψ_m and V_m) can be derived from the missile's onboard inertial navigation system (INS). The LOS angle and LOS rate (σ and $\dot{\sigma}$) can be derived from LOS gyros or the INS measurements of the seeker antenna angles. The parameters related solely to the target (Ψ_T, a_T, V_T) can not be directly measured but have to be estimated by an appropriate observer. In this paper the issue of the target parameter estimation is not considered; it is assumed that they are exactly available for the FLGL without any uncertainty.

V. Simulation Results

This section presents simulation results comparing the performance of the FLGL to the case where PN is applied for the whole intercept period with a constant effective navigation ratio. The PN guidance law is $a_m(t) = N'V_c\dot{\sigma}$. Here we consider a nonmaneuvering target traveling at a constant speed of 250 m/s with a heading of $\Psi_T - \sigma_0 = -20$ deg. The reference axis is assumed to be the direction of the initial line of sight, thus $\sigma_0 = 0$ deg. The missile is launched at R_{TM_0} with a speed of $V_m = 1000$ m/s in the heading of

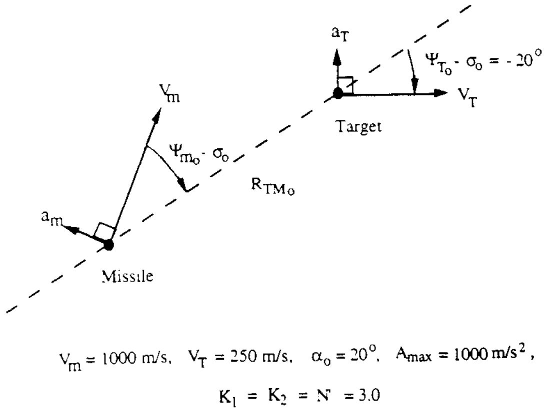

Fig. 6 Simulation geometry.

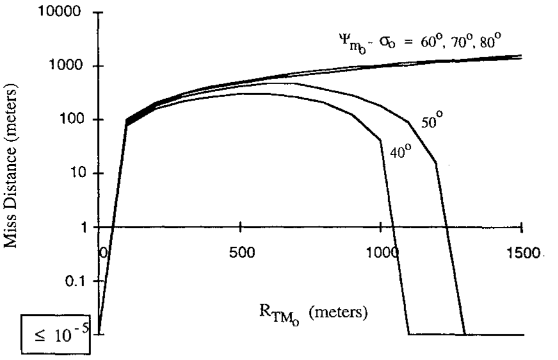

Fig. 7 Miss distance vs initial range for PN guidance.

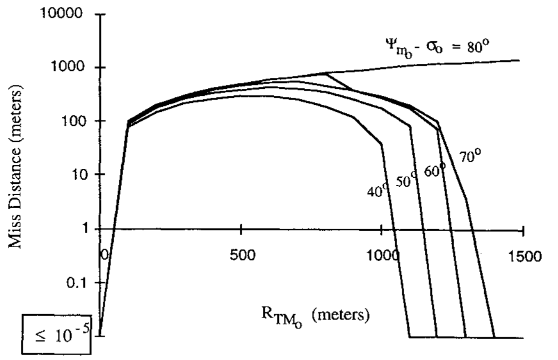

Fig. 8 Miss distance vs initial range for the FLGL.

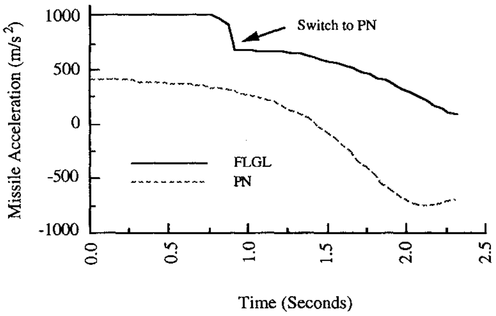

Fig. 9 Commanded missile acceleration vs time, R_{TM_0} = 1500, and $\Psi_{m_0} - \sigma_0 = -60$ deg.

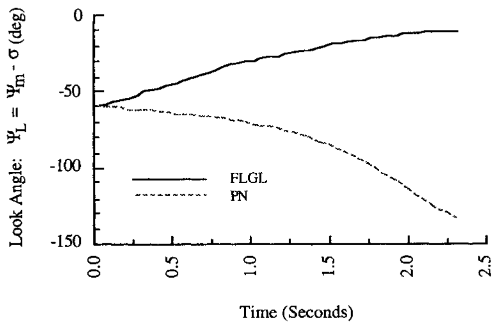

Fig. 10 Look angle ($\Psi_m - \sigma$) vs time, R_{TM_0} = 1500, and $\Psi_{m_0} - \sigma_0$ = −60 deg.

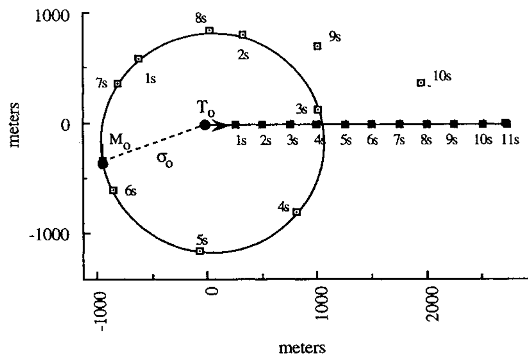

Fig. 11 FLGL missile and target position profile, R_{TM_0} = 1000, and $\Psi_{m_0} - \sigma_0$ = 80 deg.

$\Psi_{m_0} - \sigma_0$. The feedback gains chosen are $K_1 = K_2 = 3.0$, and the effective navigation ratio used is $N' = 3$. The missile's acceleration constraint is $A_{max} = 100$ g. The switching from the feedback linearizing regime to the PN regime in the FLGL is made at $\alpha_0 = 20$ deg. This initial geometry of the missile/target engagement is shown in Fig. 6.

Figures 7 and 8 show miss distance vs initial target-to-missile range R_{TM_0} for PN and FLGL, respectively. The results are presented only for $\Psi_{m_0} - \sigma_0 > 0$. When $\Psi_{m_0} - \sigma_0 < 0$ the results are similar. When the initial missile heading $|\Psi_{m_0} - \sigma_0| \leq 50$ deg, the performance of PN and the FLGL are approximately the same; both guidance laws yield miss distances $\leq 10^{-1}$ for an initial range R_{TM_0} less than 1500 m. When the initial missile heading is in a region of 50 deg $< |\Psi_{m_0} - \sigma_0| \leq 70$ deg, PN results in a miss distance that is approximately equal to R_{TM_0}, whereas FLGL gives miss distances $\leq 10^{-1}$ m for initial ranges less than 1500 m. It should be noted that in practice, when PN is used during the entire intercept period, N' is usually varied over time with N' large for large heading errors and decreased as the heading error is decreased. When the initial missile heading $|\Psi_{m_0} - \sigma_0| > 70$ deg, PN and FLGL both give miss distances approximately equal R_{TM_0}. Hence, both guidance laws lose the ability to capture the target above $|\Psi_{m_0} - \sigma_0| > 70$ deg. Simulations show that when $|\Psi_{m_0} - \sigma_0| > 70$ deg, the performance is mainly limited by the missile's acceleration constraint. In other words, it is likely that no guidance law could produce a better performance in this region. Figures 9 and 10 show typical plots of commanded missile acceleration and look angle, $\Psi_L = \Psi_m - \sigma$, vs time when the performance of PN fails to produce a small miss distance and when the performance of FLGL produces a miss distance $\leq 10^{-1}$ m. These plots are for $R_{TM_0} = 1500$ m and $\Psi_{m_0} - \sigma_0 = -60$ deg. Figure 8 demonstrates that the reason FLGL produced a smaller miss distance than PN was because FLGL commanded a larger missile acceleration during the initial stages of the missile's flight.

Figures 7 and 8 show the performance results when the simulations are terminated at the first passing, i.e., R_{TM} was minimal for the first time or when V_c changes sign. However, we can continue the flight if the missile could somehow continue to track the target. If we let the simulations continue when the first passing fails to provide a small enough miss distance, we find PN always diverges whereas FLGL will continue to guide the missile until an intercept occurs (see Figs. 11 and 12). This implies that given unlimited target tracking abilities and sufficient energy, the missile can be maneuvered to achieve any desired miss distance.

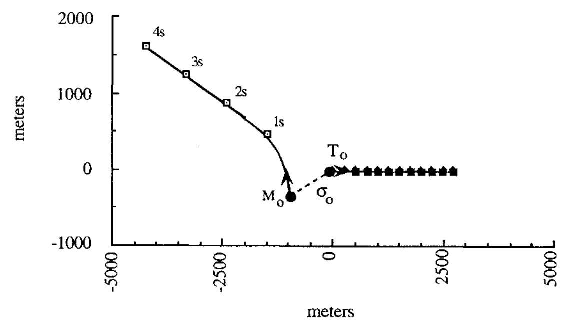

Fig. 12 PN missile and target position profile, $R_{TM_0} = 1000$, and $\Psi_{m_0} - \sigma_0 = 80$ deg.

VI. Conclusions

The feedback linearization control methodology has been applied to the guidance of a homing missile. As a result, a guidance law was derived which switches between an exact nonlinear law when the missile is far from the intercept heading to an optimal linear control law when the missile is near to the intercept heading. Simulations were presented comparing the feedback linearizing guidance law to the PN guidance law with a constant effective navigation ratio. The FLGL generally performed better when large initial heading errors were present.

Appendix A: Nonlinear Control Theory

During the past decade, there has been considerable progress in developing a systematic body of nonlinear feedback control theory. Much of this success has been a result of the meaningful generalization of fundamental concepts from geometric linear control theory. It has been demonstrated, for example, that in many common control problems involving nonlinear systems it is possible to construct a feedback control law which will transform the original nonlinear plant into a closed-loop system which is exactly linear. This feedback conversion process is called feedback linearization. Once a nonlinear plant is feedback linearized, all of the tools developed for the theory of linear systems can be directly applied to control the new closed loop.

Some fundamental concepts of nonlinear control theory for single-input, single-output systems are presented in this Appendix. The majority of the theory which follows can be found in the Refs. 18 and 19.

Single-Input, Single-Output Nonlinear Control System Description

A single-input, single-output (SISO) nonlinear system is said to be affine (linear in the control input) if it can be described in the state-space form

$$\dot{x} = f(x) + g(x)u \qquad y = h(x) \tag{A1}$$

where the state

$$x = (x_1, x_2, \ldots, x_n)^T$$

is defined on some open set U of $\Re^n$, the n-dimensional Cartesian space. The functions f, g, and h are assumed to be smooth mappings defined on U and having continuous partial derivatives of any order, i.e., C^∞ mappings. The functions f and g can be represented in the form

$$f(x) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{bmatrix}$$

$$g(x) = \begin{bmatrix} g_1(x_1, x_2, \ldots, x_n) \\ g_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ g_n(x_1, x_2, \ldots, x_n) \end{bmatrix}$$

It has been demonstrated[18,19] that when a nonlinear system of the form shown in Eq. (A1) has a well-defined relative degree, it is

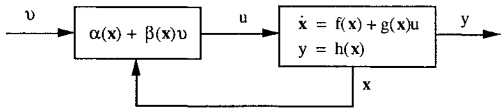

Fig. A1 Block diagram of feedback linearization.

possible to feedback linearize the system with a state feedback law of the form

$$u = \alpha(x) + \beta(x)\upsilon \tag{A2}$$

where the functions $\alpha(x)$ and $\beta(x)$ are defined on a suitable open set of U, and υ is an external reference input. A graphical depiction of the feedback linearization process is shown in Fig. A1. In effect, the linearizing feedback loop forces the mapping from υ to y to be equivalent to a chain of r integrators, where r is equivalent to the relative degree of the system. If it happens that $r = n$, then the closed-loop state-space system can be transformed via a coordinate transformation $z = \Phi(x)$ into the linear form

$$\dot{z} = Az + b\upsilon \qquad y = z_1$$

with $\{A, b\}$ being controllable. Under this condition the system is said to be fully linearizable. The details of this process are explained in the following subsections.

Relative Degree

The single-input, single-output nonlinear system described by Eq. (A1) is said to have relative degree r at a point x_0 in the state space if 1) $L_g L_f^k h(x) = 0$ for all x in a neighborhood of x_0 and all $k < r - 1$ and 2) $L_g L_f^{r-1} h(x_0) \neq 0$, where $L_g L_f^k h(x)$ denotes the Lie derivative of $L_f^k h(x)$ with respect to g. This function can be computed as follows:

$$L_g L_f^k h(x) = \sum_{i=1}^{n} \frac{L_f^k h(x)}{\partial x_i} g_i(x)$$

where

$$L_f^k h(x) = \sum_{i=1}^{n} \frac{L_f^{k-1} h(x)}{\partial x_i} f_i(x), \qquad \text{and} \qquad L_f^0 h(x) = h(x)$$

As an immediate application of the notion of relative degree, suppose that the state $x(t)$ of a given system is in a neighborhood of the state $x(t_0) = x_0$ when t is near t_0. The derivatives of the output, $y^{(k)}(t)$ for $k = 1, 2, \ldots$, can be expressed in the following manner:

$$y^{(1)}(t) = \frac{\partial h}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial h}{\partial x}\{f[x(t)] + g[x(t)]u(t)\}$$
$$= L_f h[x(t)] + L_g h[x(t)]u(t)$$

If the relative degree r is larger than 1, then by definition it follows that $L_g h[x(t)] = 0$ on some open neighborhood of x_0, yielding

$$y^{(1)}(t) = L_f h[x(t)]$$

Similarly, the second derivative of the output with respect to time is found to be

$$y^{(2)}(t) = \frac{\partial L_f h}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial L_f h}{\partial x}\{f[x(t)] + g[x(t)]u(t)\}$$
$$= L_f^2 h[x(t)] + L_g L_f h[x(t)]u(t)$$

If the relative degree of the system is greater than 2, then by definition $L_g L_f h[x(t)] = 0$ on a neighborhood of x_0, giving

$$y^{(2)}(t) = L_f^2 h[x(t)]$$

Continuing in this manner, one concludes that

$$y^{(k)}(t) = L_f^k h[x(t)]$$

for all $k < r$ and all t in a neighborhood of t_0 and

$$y^{(r)}(t_0) = L_f^r h(x_0) + L_g L_f^{r-1} h(x_0)u(t_0)$$

Thus, the relative degree r of a system is equal to the number of times the output must be differentiated to have the input appear explicitly. The preceding calculations also suggest that the functions $h(\boldsymbol{x}), L_f h(\boldsymbol{x}), \ldots, L_f^{r-1} h(\boldsymbol{x})$ have a special significance. In fact, it will be shown subsequently that these functions can be used to define a key local coordinate transformation for the system about the point $\boldsymbol{x}_0$.

Normal Forms

Given a state-space system, let a coordinate transformation be any diffeomorphism denoted by

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \Phi(\boldsymbol{x}) = \begin{bmatrix} \phi_1(\boldsymbol{x}) \\ \phi_2(\boldsymbol{x}) \\ \vdots \\ \phi_n(\boldsymbol{x}) \end{bmatrix} \tag{A3}$$

A coordinate transformation is simply a redefinition of the states of the system. In particular, suppose one is given a nonlinear system (f, g, h) with relative degree r at $\boldsymbol{x}_0$. Let the first r functions of a coordinate transformation be defined as

$$\begin{bmatrix} \phi_1(\boldsymbol{x}) \\ \phi_2(\boldsymbol{x}) \\ \vdots \\ \phi_r(\boldsymbol{x}) \end{bmatrix} \triangleq \begin{bmatrix} h(\boldsymbol{x}) \\ L_f h(\boldsymbol{x}) \\ \vdots \\ L_f^{r-1} h(\boldsymbol{x}) \end{bmatrix} \tag{A4}$$

If $r < n$, it is well known[18] that we can find $n-r$ additional functions such that the transformation

$$z = \begin{bmatrix} z_1(\boldsymbol{x}) \\ z_2(\boldsymbol{x}) \\ \vdots \\ z_r(\boldsymbol{x}) \\ z_{r+1}(\boldsymbol{x}) \\ \vdots \\ z_n(\boldsymbol{x}) \end{bmatrix} = \Phi(\boldsymbol{x}) = \begin{bmatrix} \phi_1(\boldsymbol{x}) \\ \phi_2(\boldsymbol{x}) \\ \vdots \\ \phi_r(\boldsymbol{x}) \\ \phi_{r+1}(\boldsymbol{x}) \\ \vdots \\ \phi_n(\boldsymbol{x}) \end{bmatrix} \tag{A5}$$

defines a valid local coordinate transformation for some neighborhood of $\boldsymbol{x}_0$. In particular, it is always possible to choose $\phi_{r+1}(\boldsymbol{x}), \ldots, \phi_n(\boldsymbol{x})$ such that

$$L_g \phi_i(\boldsymbol{x}) = 0 \tag{A6}$$

for all $r + 1 \leq i \leq n$ and all $\boldsymbol{x}$ near $\boldsymbol{x}_0$.

The original nonlinear system can then be expressed in the new coordinate frame via the following process:

$$\begin{aligned} \frac{\mathrm{d}z_1}{\mathrm{d}t} &= \frac{\partial \phi_1}{\partial \boldsymbol{x}} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \frac{\partial h(\boldsymbol{x})}{\partial \boldsymbol{x}} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \frac{\partial h(\boldsymbol{x})}{\partial \boldsymbol{x}} \{f[\boldsymbol{x}(t)] + g[\boldsymbol{x}(t)]u(t)\} \\ &= \frac{\partial h(\boldsymbol{x})}{\partial \boldsymbol{x}} f[\boldsymbol{x}(t)] + \frac{\partial h(\boldsymbol{x})}{\partial \boldsymbol{x}} g[\boldsymbol{x}(t)]u(t) \\ &= L_f h[\boldsymbol{x}(t)] + L_g h[\boldsymbol{x}(t)]u(t) \end{aligned}$$

If the relative degree r is larger than 1, then by definition $L_g h[\boldsymbol{x}(t)] = 0$ and, hence,

$$\begin{aligned} \frac{\mathrm{d}z_1}{\mathrm{d}t} &= L_f h[\boldsymbol{x}(t)] \\ &= \phi_2[\boldsymbol{x}(t)] \\ &= z_2(t) \end{aligned}$$

Similarly, continuing until z_{r-1}

$$\begin{aligned} \frac{\mathrm{d}z_{r-1}}{\mathrm{d}t} &= \frac{\partial \phi_{r-1}}{\partial \boldsymbol{x}} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \frac{\partial (L_f^{r-2} h)}{\partial \boldsymbol{x}} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \\ &= L_f^{r-1} h[\boldsymbol{x}(t)] = \phi_r[\boldsymbol{x}(t)] = z_r(t) \end{aligned}$$

For z_r it follows that

$$\frac{\mathrm{d}z_r}{\mathrm{d}t} = L_f^r h[\boldsymbol{x}(t)] + L_g L_f^{r-1} h[\boldsymbol{x}(t)]u(t)$$

Define

$$a(\boldsymbol{x}) = L_g L_f^{r-1} h[\boldsymbol{x}(t)]$$

$$b(\boldsymbol{x}) = L_f^r h[\boldsymbol{x}(t)]$$

and substitute the x coordinates in the preceding equations by $\boldsymbol{x} = \Phi^{-1}(z)$ to obtain

$$\frac{\mathrm{d}z_r}{\mathrm{d}t} = b \circ \Phi^{-1}(z) + [a \circ \Phi^{-1}(z)]u(t)$$

near the point $z_0 = \Phi(\boldsymbol{x}_0)$. By the second condition in the definition of relative degree the coefficient $[a \circ \Phi^{-1}(z)]$ is nonzero for all z in a neighborhood of z_0.

With respect to the remaining coordinates, one cannot necessarily expect any special structure for their corresponding equations. However, if $\phi_{r+1}(\boldsymbol{x}), \ldots, \phi_n(\boldsymbol{x})$ are chosen such that Eq. (A6) is satisfied, then

$$\begin{aligned} \frac{\mathrm{d}z_i}{\mathrm{d}t} &= \frac{\partial \phi_i}{\partial \boldsymbol{x}} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \frac{\partial \phi_i}{\partial \boldsymbol{x}} \{f[\boldsymbol{x}(t)] + g[\boldsymbol{x}(t)]u(t)\} \\ &= L_f \phi_i[\boldsymbol{x}(t)] + L_g \phi_i[\boldsymbol{x}(t)]u(t) \\ &= L_f \phi_i[\boldsymbol{x}(t)] \qquad \text{for} \qquad i = r+1, \ldots, n \end{aligned}$$

Setting

$$q_i(z) = L_f \phi_i[\Phi^{-1}(z)] \qquad \text{for} \qquad i = r+1, \ldots, n$$

one can express the preceding equations more compactly as

$$\frac{\mathrm{d}zi_i}{\mathrm{d}t} = q_i[z(t)]$$

Thus using the coordinate transformation defined by Eqs. (A4) and (A5) the nonlinear system in the new coordinates assumes the form

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= b(z) + a(z)u \\ \dot{z}_{r+1} &= q_{r+1}(z) \\ &\vdots \\ \dot{z}_n &= q_n(z) \end{aligned} \tag{A7}$$

with the output function in the new coordinate system given by

$$y = z_1$$

The system given by Eq. (A7) is said to be in normal form. Figure A2 shows an illustration of this normal form structure.

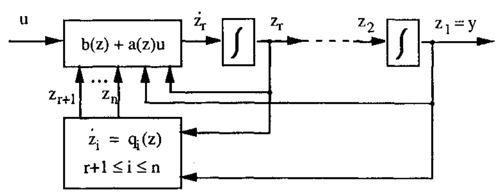

Fig. A2 Diagram of system's normal form structure.

Feedback Linearization

For the nonlinear system defined by Eq. (A1), consider the state feedback rule of the form

$$u = \alpha(x) + \beta(x)v$$

where v is an external reference input, and $\alpha(x)$ and $\beta(x)$ are real-valued functions defined on a suitable open set of U. After application of this state feedback law, the closed-loop system becomes

$$\dot{x} = f(x) + g(x)\alpha(x) + g(x)\beta(x)v \qquad y = h(x)$$

which was illustrated in Fig. A1. The feedback linearization problem is to determine a feedback rule $u = \alpha(x) + \beta(x)v$ such that the input-output system from v to y is a linear mapping. First, consider a system which has full relative degree, i.e., $r = n$. The system in normal form is

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= b(z) + a(z)u \\ y &= z_1 \end{aligned} \tag{A8}$$

Now suppose the state feedback control law

$$u = \frac{1}{a(z)}[-b(z) + v] \tag{A9}$$

is applied, which is well defined near $z_0 = \Phi(x_0)$ since $a(z) \neq 0$. Substituting Eq. (A9) into Eq. (A8) results in a closed-loop system of the form

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= v \\ y &= z_1 \end{aligned}$$

which is clearly a linear and controllable system with transfer function $H(s) = s^{-n}$. Thus, we conclude that any nonlinear system with relative degree n at some point x_0 can be transformed into a system which is linear and controllable in a neighborhood of the point $z_0 = \Phi(x_0)$. Figure A3 shows the structure of the system after the coordinate change and linearizing feedback have been applied. The expression of the linearizing feedback law, Eq. (A8), in terms of the original coordinate frame is as follows:

$$u = \frac{1}{L_g L_f^{n-1} h(x)}\left[- L_f^n h(x) + v\right] \tag{A10}$$

If the system has a relative degree $r < n$, the state feedback Eq. (A9) will produce a state-space system which is only partially linear. In particular, substitute Eq. (A9) into Eq. (A7) to obtain

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= v \\ \dot{z}_{r+1} &= q_{r+1}(z) \\ &\vdots \\ \dot{z}_n &= q_n(z) \\ y &= z_1 \end{aligned} \tag{A11}$$

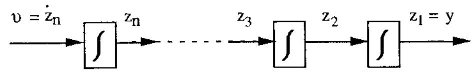

Fig. A3 Fully feedback linearized system expressed in normal form.

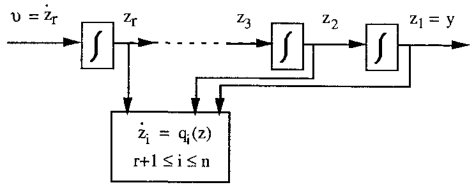

Fig. A4 Feedback linearized system in normal form when $r < n$.

The resulting feedback linearized system will consist of a linear state-space subsystem having dimension r and an unobservable (possibly) nonlinear subsystem of dimension $n - r$. In this case the expression of the linearizing feedback, Eq. (A8), in terms of the original coordinate frame is

$$u = \frac{1}{L_g L_f^{r-1} h(x)}\left[- L_f^r h(x) + v\right] \tag{A12}$$

Figure A4 shows the structure of the closed-loop system given by Eq. (A11) and Eq. (A12). Observe that the input-output mapping is linear and described by the transfer function $H(s) = s^{-r}$.

Zero Dynamics

For systems with relative degree $r < n$, Fig. A4 shows there exists a subsystem, defined by $\dot{z}_i = q_i(z)$, $r + 1 \leq i \leq n$, whose behavior does not affect the output. Partition the state vector in normal form as

$$\xi = \begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix}, \qquad \eta = \begin{bmatrix} z_{r+1} \\ \vdots \\ z_n \end{bmatrix}$$

Then the normal form equations can be expressed as

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= b(\xi, \eta) + a(\xi, \eta)u \\ \dot{\eta} &= q(\xi, \eta) \\ y &= z_1 \end{aligned}$$

The problem of zeroing the output consists of determining an initial state x_0 and an input function $u_0(t)$ such that the corresponding output of the system $y(t)$ is identically zero for t near $t_0 = 0$. Since the output of the system is $y(t) = z_1(t)$, one can see that this implies that $\xi(t) = 0$ for t near $t_0 = 0$ and, thus, $\dot{z}_1(t) = \dot{z}_2(t) = \cdots = \dot{z}_r(t) = 0$ for t near $t_0 = 0$. Hence, when the output of the system is identically the zero function, its state is constrained to evolve in such a way that $\xi(t)$ is exactly the zero function. In addition, the input $u(t)$ must necessarily be the unique solution of the equation

$$0 = b[0, \eta(t)] + a[0, \eta(t)]u(t)$$

where $\eta(t)$ has a trajectory governed by the differential equation

$$\dot{\eta}(t) = q[0, \eta(t)] \tag{A13}$$

with initial condition $\eta(0) = \eta_0$. These dynamics describing the internal behavior of the system when input and initial conditions have been chosen to constrain the output to remain identically zero are called the zero dynamics of the system. This is a nonlinear generalization of the well-known transmission-blocking property of zeros for linear time-invariant systems.

Appendix B: Derivation of the Feedback Linearizing Law for the Missile Guidance Problem

The missile's and target's equations of motion, Eqs. (1–5), give a representation of the guidance system in terms of a set of nonlinear differential equations which are affine in the control variable a_m. Hence, the guidance system can be modeled as a eighth-order state-space system of the form

$$\dot{x} = f(x, a_T) + g(x)a_m \qquad y = h(x) \tag{B1}$$

where x is the system's state vector. Specifically, the eighth-order state-space description of the guidance system in Cartesian coordinates is found to be

$$\frac{d}{dt}\begin{bmatrix} x_m \\ y_m \\ V_{x_m} \\ V_{y_m} \\ x_T \\ y_T \\ V_{x_T} \\ V_{y_T} \end{bmatrix} = \begin{bmatrix} V_{x_m} \\ V_{y_m} \\ 0 \\ 0 \\ V_{x_T} \\ V_{y_T} \\ -a_T \sin(\Psi_T) \\ a_T \cos(\Psi_T) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sin(\Psi_m) \\ \cos(\Psi_m) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} a_m \tag{B2}$$

$$R_{\mathrm{TM}} = \sqrt{(x_m - x_T)^2 + (y_m - y_T)^2}$$

where

$$\Psi_m = \tan^{-1}\frac{V_{y_m}}{V_{x_m}}, \qquad \text{and} \qquad \Psi_T = \tan^{-1}\frac{V_{y_T}}{V_{x_T}}$$

Defining

$$x_R = x_m - x_T$$

$$y_R = y_m - y_T$$

$$V_{x_R} = V_{x_m} - V_{x_T}$$

$$V_{y_R} = V_{y_m} - V_{y_T}$$

the Lie derivatives are computed as follows:

$$h(x) = \sqrt{x_R^2 + y_R^2} = R_{\mathrm{TM}}$$

$$L_g h(x) = 0, \qquad \text{for all } x$$

$$L_f h(x) = \frac{x_R V_{x_R} + y_R V_{y_R}}{\sqrt{x_R^2 + y_R^2}} = \dot{R}_{\mathrm{TM}}$$

$$L_g L_f h(x) = \frac{y_R \cos(\Psi_m) - x_R \sin(\Psi_m)}{\sqrt{x_R^2 + y_R^2}} = \sin(\Psi_m - \sigma)$$

$$L_f^2 h(x) = \frac{(x_R V_{y_R} - V_{x_R} y_R)^2}{R_{\mathrm{TM}}^3} + \left(\frac{x_R}{R_{\mathrm{TM}}}\sin(\Psi_T) - \frac{y_R}{R_{\mathrm{TM}}}\cos(\Psi_T)\right)a_T = R_{\mathrm{TM}}\dot{\sigma}^2 - a_T \sin(\Psi_T - \sigma)$$

Note that $L_g L_f h(x) \neq 0$, except when the missile's velocity vector is pointed directly at the target. Therefore, the system does not have a well-defined relative degree when the missile's velocity is directly along the line of sight. However, for all other points in the state space the system has a relative degree equal to two.

The linearizing feedback is calculated from the standard formula (A12)

$$a_m = \frac{1}{L_g L_f h(x)}\left[\upsilon - L_f^2 h(x)\right]$$

$$a_m = \frac{\sqrt{x_R^2 + y_R^2}}{y_R \cos(\Psi_m) - x_R \sin(\Psi_m)}\left[\upsilon - \frac{(x_R V_{y_R} - V_{x_R} y_R)^2}{R_{\mathrm{TM}}^3} - \left(\frac{x_R}{R_{\mathrm{TM}}}\sin(\Psi_T) - \frac{y_R}{R_{\mathrm{TM}}}\cos(\Psi_T)\right)a_T\right]$$

$$a_m = \frac{1}{\sin(\Psi_m - \sigma)}\left[\upsilon - R_{\mathrm{TM}}\dot{\sigma}^2 + a_T \sin(\Psi_T - \sigma)\right]$$

The linear system obtained after the feedback is applied is

$$\dot{R} = -V_c \qquad \ddot{R} = \upsilon$$

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